Compact automorphism groups of vertex operator algebras

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Abstract

Let V be a simple vertex operator algebra which admits the continuous, faithful action of a compact Lie group G of automorphisms. We establish a Schur-Weyl type duality between the unitary, irreducible modules for G and the irreducible modules for V^G which are contained in V where V^G is the space of G-invariants of V. We also prove a concomitant Galois correspondence between vertex operator subalgebras of V which contain V^G and closed Lie subgroups of G in the case that G is abelian. These results extend those of [DM1] and [DM2].

1 Introduction

In this paper we study simple vertex operator algebras V which admit a continuous action of a compact Lie group G (possibly finite). This is a natural continuation of our earlier work [DM1], [DM2] in which we limited ourselves to the case of finite solvable groups. Such pairs (V, G) occur naturally: for example the vertex operator algebras V_L associated with a positive definite even lattice L (cf. [B], [FLM]) admit Lie groups which arise from exponentiating the Lie algebra associated with L. A special case of this is when L is itself a root lattice (of type ADE) associated to a finite-dimensional simple Lie algebra $\mathfrak{g}$, in which case V_L is the basic representation of the affine Lie algebra $\hat{\mathfrak{g}}$ and V_L admits a unitary action of the corresponding Lie group. This latter situation was studied recently in a preprint of Baker and Tamanoi [BT], in which they carried out some explicit calculations and made a general conjecture (which we explain in due course). Our main theorem yields, as a very special case, a proof of their conjecture. It also yields an extension of Theorem 3 of [DM1], originally proved only for solvable groups, to *all* finite groups.

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The main result, Theorem 2.4 below, is a duality theorem of Schur-Weyl type. Precisely, it says the following: let I index the finite dimensional irreducible (continuous) representations of G, and for $\lambda \in I$ let W_λ be the corresponding G-module. Then there is a decomposition of V, considered as $G \times V^G$-module, as follows:

$$(*) \qquad V = \bigoplus_{\lambda \in I} W_\lambda \otimes V_\lambda$$

where each V_λ is a *non-zero, irreducible* V^G-module. Here, V^G is the vertex operator subalgebra of V consisting of the G-invariants. Moreover, V_λ and V_μ are *inequivalent* V^G-modules if $\lambda \neq \mu$. Such results have been predicted – at least for finite groups – in the physics literature [DVVV]. The decomposition $(*)$ was obtained for finite solvable groups in [DM1]. The conjecture of [BT] is essentially the decomposition $(*)$ in the case $V = V_L$ with L of type A_l and $G \simeq SU(l+1)$. Note in particular that the Theorem implies that V^G is itself a simple vertex operator algebra, since V^G is just the subspace V_1 corresponding to the trivial G-module. We also give another kind of duality or reciprocity result (Corollary 2.7) which compares the decomposition $(*)$ for a group G and subgroup H. Thus the pair (G, V^G) behaves just like a classical dual pair [H2].

We also give an infinitesimal version of $(*)$ (Theorem 2.8). Namely, we define the notion of a *derivation* of a vertex operator algebra and observe that derivations form a Lie algebra. Then the analogue of $(*)$ holds where I is now the set of dominant weights for a finite-dimensional semi-simple Lie algebra of derivations of V and W_λ is the corresponding irreducible highest weight module.

As in [DM1], [DM2], $(*)$ also has applications to what we have called *quantum Galois theory* (loc. cit.). Precisely, one is looking for a Galois correspondence between vertex operator subalgebras of V which contain V^G and subgroups of G. The case in which G is finite and either nilpotent or dihedral was discussed in (loc. cit.), where the exact analogue of the usual Galois correspondence was established. Here we consider the case of compact abelian Lie groups (we could easily extend this to the case of compact nilpotent Lie groups, along the line of Theorem 2 of [DM2]); for infinite groups one expects only "closed" subgroups of G to intervene in the Galois correspondence, and we shall see that this is so. Namely, we prove that if G is a compact abelian Lie group acting continuously on the simple vertex operator algebra V, then the map $H \mapsto V^H$ gives a bijection between the *closed Lie subgroups* of G and the vertex operator subalgebras of V which contain V^G. For example, if $G = S^1$ is the circle then the vertex operator subalgebras of V which contain V^{S^1} are, apart from V^{S^1} itself, precisely these of the form V^F for a finite (cyclic) subgroup F of S^1. Thus in this case the "Krull topology" one puts on G is the fc (closed=finite) topology.

The method of proof that we use is rather different from that of [DM1], [DM2], though some key points are the same. We employ a certain $\mathbb{Z}$-graded Lie algebra $\hat{V}$ naturally associated to V that we have used elsewhere [DLM], combined with some of Howe's ideas [H1] which we have adapted to the present situation.

The main theorem and its infinitesimal analogue are proved in Section 2. The Galois correspondence is proved in Section 3.

2   The main theorem

Let $V = (V, Y, \mathbf{1}, \omega)$ be a simple vertex operator algebra and G a group of automorphisms of V (cf. [FLM]). Assume that G is a finite-dimensional compact Lie group and that the action of G is continuous. Let $\{W_\lambda | \lambda \in I\}$ be the set of inequivalent finite-dimensional irreducible representations. Then any finite dimensional representation of G is a direct sum of W_λ's. Since each homogeneous subspace V_n of V is finite dimensional, V is a direct sum of finite-dimensional irreducible G-modules:

$$V = \bigoplus_{\lambda \in I} V^\lambda$$

where V^λ is the sum of all G-submodules of V which are isomorphic to W_λ. In particular, the subspace of G-invariants V^G is exactly V^1 if W_1 is the 1-dimensional trivial representation.

Consider the quotient space

$$\hat{V} = \mathbb{C}[t, t^{-1}] \otimes V / D\mathbb{C}[t, t^{-1}] \otimes V$$

where $D = \frac{d}{dt} \otimes 1 + 1 \otimes L(-1)$. Denote by $v(n)$ the image of $v \otimes t^n$ in $\hat{V}$ for $v \in V$ and $n \in \mathbb{Z}$. Then $\hat{V}$ is $\mathbb{Z}$-graded by defining the degree of $v(n)$ to be $\text{wt} v - n - 1$ if v is homogeneous. Denote the homogeneous subspace of degree n by $\hat{V}(n)$. The space $\hat{V}$ is, in fact, a $\mathbb{Z}$-graded Lie algebra with bracket

$$[a(m), b(n)] = \sum_{i=0}^{\infty} \binom{m}{i} a_i b(m + n - i)$$

(see [L1] and [DLM]). We define a G-action on $\hat{V}$ by $gv(n) = (gv)(n)$. Then $\hat{V}(n)$ is isomorphic to V as G-modules. In particular, $\hat{V}$ is a semi-simple G-module and $(\hat{V})^G = \hat{V}^G$. The following proposition is proved in [L1] and [DM1]:

Proposition 2.1 V *is spanned by* $\{u_n v | u \in V, n \in \mathbb{Z}\}$ *where* $v \in V$ *is a fixed nonzero vector.*

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For any $n \in \mathbb{Z}$ we define $V_{\leq n} = \sum_{m \leq n} V_m$ and $V_{>n} = \sum_{m>n} V_m$. Then $V = V_{\leq n} \oplus V_{>n}$ and $\mathrm{End}V = \mathrm{End}V_{\leq n} \oplus \mathrm{End}V_{>n} \oplus \mathrm{Hom}_{\mathbb{C}}(V_{\leq n}, V_{>n}) \oplus \mathrm{Hom}_{\mathbb{C}}(V_{>n}, V_{\leq n})$. Note that $\mathrm{End}V$ is a G-module via conjugation and that $\mathrm{End}V_{\leq n}$ is a semi-simple submodule as $V_{\leq n}$ is finite-dimensional. Let T' denote the projection of $T \in \mathrm{End}V$ onto $\mathrm{End}V_{\leq n}$. Then this projection is a G-module homomorphism.

Since $gY(u, z)g^{-1} = Y(gu, z)$ for $v \in V$ and $g \in G$ we see that the map $v(m) \mapsto v_m$ is a G-module homomorphism from $\hat{V}$ to $\mathrm{End}V$. Moreover, $f \colon v(n) \mapsto v'_n$ is a G-module homomorphism from $\hat{V}$ to $\mathrm{End}V_{\leq n}$ for any n as g preserves both $V_{\leq n}$ and $V_{>n}$. The following lemma is an immediate consequence of Proposition 2.1.

Lemma 2.2 *The map f is an epimorphism from the semi-simple G-module $\hat{V}$ to the semi-simple G-module $\mathrm{End}V_{\leq n}$ for any n. In particular, $f(\hat{V}^G) = (\mathrm{End}V_{\leq n})^G$.*

Fix $\lambda \in I$. As the action of G on W_λ induces an epimorphism $\mathbb{C}[G] \to \mathrm{End}W_\lambda$, we can define the space of highest weight vectors in W_λ as follows: fix a basis of W_λ, with e_{ij} the corresponding standard basis of $\mathrm{End}W_\lambda$. Then $w \in W_\lambda$ is a highest weight vector if $e_{ij}w = 0$ whenever $j > i$ or $i = j, i > 1$, and if $e_{11}w = w$. We can then define the space V_λ of highest weight vectors in V^λ in the obvious way.

Lemma 2.3 *Each V_λ is a module for V^G.*

Proof: Since the action of G commutes with the action of vertex operator $Y(u, z)$ for $u \in V^G$, the lemma follows immediately. $\square$

Note that as both G-module and V^G-module, V^λ and $W_\lambda \otimes V_\lambda$ are isomorphic. Here, G acts on the first tensor factor of $W_\lambda \otimes V_\lambda$ and V^G acts on the second tensor factor. The main result of this paper is the following.

Theorem 2.4 *(1) Each V_λ is an irreducible V^G-module.*
(2) V^λ is non-zero for any $\lambda \in I$.
(3) V_λ and V_μ are isomorphic V^G-modules if, and only if, $\lambda = \mu$.

Proof: The main idea in the proof of (1) and (3) goes back to Howe's theory of dual pairs (see Theorem 8 of [H1]).

For (1) it is sufficient to show that as a V^G-module, V_λ is generated by any nonzero vector of V_λ. First choose $x, y \in V_\lambda$ to be homogeneous and linearly independent and $n \in \mathbb{Z}$ such that $x, y \in V_{\leq n}$. Then as a G-module we can write $V_{\leq n}$ as a direct sum of three submodules $W_\lambda \otimes x \oplus W_\lambda \otimes y \oplus W$. Define a map $\alpha \in \mathrm{End}V_{\leq n}$ such that $\alpha(u \otimes x + v \otimes y + w) = v \otimes x + u \otimes y + w$ for $u, v \in W_\lambda$ and $w \in W$. Then it is easy to see that $g\alpha g^{-1} = \alpha$ for all $g \in G$. That is, $\alpha \in (\mathrm{End}V)^G$.

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On the other hand by Lemma 2.2 there exist homogeneous vectors $v^1, ..., v^k \in V^G$ and $m_1, ..., m_k \in \mathbb{Z}$ such that $(\sum_{i=1}^{k} v_{m_i}^i)' = \alpha$. Thus $(\sum_{i=1}^{k} v_{m_i}^i)'x = y$. Since x, y and v_i are homogeneous there exists a subset J of $\{1, ..., k\}$ such that $\sum_{i \in J} v_{m_i}^i x = y$. That is, V_λ is generated by any nonzero homogeneous vector of V_λ. If x is not homogeneous we can use the operator $L(0)$ to produce a nonzero homogeneous vector $x' \in V_\lambda$ and the argument above gives (1).

For (2) we first assert that there exists a faithful G-module W such that $W = W_{\lambda_1} + \cdots + W_{\lambda_k}$ with $V^{\lambda_i} \neq 0$. Set $K_n = \ker(G, V_{\leq n})$. Then $\cdots K_n \supset K_{n+1} \cdots$ is a descending chain of closed subgroups of G. Since G is a finite-dimensional Lie group, there exists $N \geq 0$ such that $\dim K_n$ is a constant if $n \geq N$. Let K_n^0 be the connected component of K_n which contains the identity e. Then $K_n^0 / K_n^0 \cap K_{n+1} \simeq K_n^0 K_{n+1} / K_{n+1} \subset K_n / K_{n+1}$. As $K_n^0 / K_n^0 \cap K_{n+1}$ is connected and compact and K_n / K_{n+1} is a finite group if $n \geq N$, $K_n^0 / K_n^0 \cap K_{n+1}$ is a one point set. That is, $K_n^0 = K_{n+1}^0$. This implies that K_N^0 acts trivially on V. Thus $K_n^0 = e$ if $n \geq N$ and K_n is finite. Consequently, $K_n = e$ and the representation of G on $V_{\leq n}$ is faithful if n is big enough. For such n we may take $W = W_{\lambda_1} + \cdots + W_{\lambda_k}$ where $W_{\lambda_1}, ..., W_{\lambda_k}$ are all the distinct W_λ's contained in $V_{\leq n}$. Our assertion follows.

Let $\Lambda = \{\lambda \in I | V^\lambda \neq 0\}$. It was proved in [DM1] that if G is a finite group and $\lambda_1, \lambda_2 \in \Lambda$ and W_γ for $\gamma \in I$ is a submodule of $W_{\lambda_1} \otimes W_{\lambda_2}$ then $\gamma \in \Lambda$ (see the proof of Theorem 2 of [DM1]). The same result holds here with the same proof. We refer the reader to [DM1] for details.

Next we show that if $\lambda \in \Lambda$ then $\lambda^* \in \Lambda$ where W_{λ^*} is the conjugate representation. By Proposition 2.1 V is spanned by $\{u_n W_\lambda | u \in V, n \in \mathbb{Z}\}$. For fix $n \in \mathbb{Z}$ and $\mu \in \Lambda$ observe that the span of $\{u_n W_\lambda | u \in W_\mu\}$ is a G-submodule of $W_\lambda \otimes W_\mu$. Note that the trivial G-module necessarily occurs in V, and that the trivial module is a submodule of $W_\lambda \otimes W_\mu$ if, and only if, $\nu = \lambda^*$. So indeed we have $\lambda^* \in \Lambda$ whenever $\lambda \in \Lambda$.

It is well-known that every irreducible module is contained in $M^{\otimes m} \otimes \bar{M}^{\otimes n}$ for some positive integers m, n whenever M is a faithful representation of G and $\bar{M}$ is its conjugate (eg, Theorem 4.4 of [BD]). Now W is a faithful representation of G inside V and its conjugate $\bar{W} = \sum W_{\lambda_i^*}$ also lies in V. Thus $\Lambda = I$. This proves (2).

The proof of (3) is similar to that of (1). We first take n such that $V_n^\lambda \neq 0$, as we may by (2). Then $V_{\leq n}$ is a direct sum of three G-modules $V_n^\lambda \oplus V_n^\mu \oplus M$ for a suitable G-submodule M of $V_{\leq n}$. Define $\beta \in \mathrm{End} V_{\leq n}$ such that it is the identity on V_n^λ and zero on V_n^μ and M. Clearly $\beta \in (\mathrm{End} V_{\leq n})^G$. Use Lemma 2.2 to obtain homogeneous vectors $v^1, ..., v^k$ of V^G such that $\sum v_{\mathrm{wt} v^i - 1}^i$ is the identity on V_n^λ and zero on V_n^μ and M. Thus there is no V^G-module homomorphism between V_λ and V_μ, as required. $\square$

Corollary 2.5 *Let V be a simple vertex operator algebra and G be a finite automorphism group of V. Then*

$$V = \sum_{\chi \in \hat{G}} W_\chi \otimes V_\chi$$

where W_χ is an irreducible G-module affording χ and V_χ is an irreducible V^G-module. Moreover, V_χ and V_ψ are non-zero inequivalent V^G-modules if χ and ψ are different irreducible characters of G.

Remark 2.6 *Corollary 2.5 was proved in [DM1] in the case that G is a finite solvable group (loc. cit., Theorems 2 and 3).*

There is a reciprocity law attached to the dual pair (G, V^G) as in the classical case (cf. [H2]). Let H be a compact Lie subgroup of G. Then by Theorem 2.4 there is a decomposition of V into irreducible $H \times V^H$-modules

$$V = \bigoplus_{\mu \in J} X_\mu \otimes V_\mu$$

where J indexes the finite dimensional irreducible (continuous) representations of H, X_μ is the corresponding H-module for $\mu \in J$, and V_μ is the corresponding irreducible V^H-module.

Let us decompose the representations of G into a direct sum of H-modules:

$$W_\lambda = \bigoplus_{\mu \in J} m_{\lambda\mu} X_\mu$$

for $\lambda \in I$ where $m_{\lambda\mu}$ is the multiplicity of μ occurring in λ. By Theorem 2.4, V is a complete reducible V^G-module. Theorefore, the V^H-modules V_μ for $\mu \in J$ decompose into a direct sum of irreducible V^G-modules:

$$V_\mu = \bigoplus_{\lambda \in I} n_{\mu\lambda} V_\lambda.$$

Note that both $m_{\lambda\mu} = \dim \operatorname{Hom}_H(W_\lambda, X_\mu)$ and $n_{\mu\lambda} = \dim \operatorname{Hom}_{V^G}(V_\mu, V_\lambda)$ are finite.

Corollary 2.7 *The following reciprocity holds:*

$$m_{\lambda\mu} = n_{\mu\lambda}$$

for all $\lambda \in I$ and $\mu \in J$.

Proof: The proof is exactly the same as that in Section 5.7 of [H2] in the classical case. For completeness we repeat this short proof.

Note that V is a complete reducible $H \times V^G$-module and consider the decomposition of V as a $H \times V^G$-module. Let $\lambda \in I$ and $\mu \in J$. Clearly, in the decomposition of V into $H \times V^H$-modules, $X_\mu \otimes V_\lambda$ only occurs in $X_\mu \otimes V_\mu$ with multiplicity $n_{\mu\lambda}$. Similarly, in the decomposition of V into $G \times V^G$-modules, $X_\mu \otimes V_\lambda$ only occurs in $W_\lambda \otimes V_\lambda$ with multiplicity $n_{\mu\lambda}$. Thus $m_{\lambda\mu} = n_{\mu\lambda}$, as claimed. $\square$

Next we present a Lie algebra version of Theorem 2.4. We need the concept of a derivation of a vertex operator algebra. A derivation D of vertex operator algebra V is a degree-preserving linear map of V such that $D\mathbf{1} = D\omega = 0$ and $[D, Y(u,z)] = Y(Du,z)$ for all $u \in V$. Denote the set of all derivations of V by $\mathrm{Der}V$. Then $\mathrm{Der}V$ is a Lie algebra. Let $\mathfrak{g}$ be a semi-simple Lie subalgebra of $\mathrm{Der}V$. Then V is completely reducible $\mathfrak{g}$-module: $V = \oplus_{\lambda \in P} V^\lambda$ where P is the set of dominant weights of $\mathfrak{g}$ and V^λ is the sum of all irreducible $\mathfrak{g}$-modules which are isomorphic to the highest weight module $L(\lambda)$ for $\mathfrak{g}$ with highest weight λ. Let V_λ be the space of all highest weight vectors in V^λ. Then $V^0 = V_0 = V^{\mathfrak{g}}$ is the space of $\mathfrak{g}$-invariants of V. That is, $V^{\mathfrak{g}} = \{v \in V | \mathfrak{g}v = 0\}$. It is easy to show that each V_λ is a module for $V^{\mathfrak{g}}$. Then $V^\lambda = L(\lambda) \otimes V_\lambda$ as $\mathfrak{g} \otimes V$-modules where $\mathfrak{g}$ acts on the first tensor factor and $V^{\mathfrak{g}}$ acts on the second tensor factor. Now use the proof of Theorem 2.4 to get

Theorem 2.8 V^λ is non-zero for all $\lambda \in P$, and V_λ is an irreducible $V^{\mathfrak{g}}$-module. Moreover, V_λ and V_μ are isomorphic $V^{\mathfrak{g}}$-modules if, and only, if $\lambda = \mu$. In particular, $V^{\mathfrak{g}}$ is a simple vertex operator algebra.

Now we apply this result to the vertex operator algebras associated to highest weight representations of affine Lie algebras. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra and $\hat{\mathfrak{g}}$ be the corresponding affine Lie algebra. Denote by $L(\ell, \lambda)$ the highest weight $\hat{\mathfrak{g}}$-module of level ℓ (which is a fixed positive integer) with highest weight $\lambda \in P$. It is well known (cf. [DL], [FZ], [L2]) that $L(\ell, 0)$ is a simple vertex operator algebra. It is clear that $\mathfrak{g}$ acts on $L(\ell, 0)$ as derivations. The following corollary is an immediate consequence of Theorem 2.8.

Corollary 2.9 $L(\ell, 0)$ has the following decomposition:

$$L(\ell, 0) = \bigoplus_{\lambda \in P} L(\lambda) \otimes L(\ell, 0)_\lambda$$

where each $L(\ell, 0)_\lambda$ is an irreducible $L(\ell, 0)^{\mathfrak{g}}$-module and $L(\ell, 0)_\lambda$ and $L(\ell, 0)_\mu$ are isomorphic $L(\ell, 0)^{\mathfrak{g}}$-modules if, and only, if $\lambda = \mu$.

3 Galois correspondence

We will prove

Theorem 3.1 *Let V be a simple vertex operator algebra and let G be a compact abelian Lie group of (faithful , continuous) automorphisms of V. Then the correspondence*

$$H \to V^H$$

sets up a bijection between the closed Lie subgroups of G and the vertex operator subalgebras of V which contain V^G.

With the main Theorem 2.4 now available, the proof follows the lines of the corresponding result (Theorem 1 of [DM1]) for finite abelian groups.

We know [BD] that G is isomorphic to $A \times T^n$ where A is a finite abelian group and T^n the n-torus. And Pontryagin duality says that the dual group $\hat{G}$ of unitary characters is isomorphic to $\hat{A} \times \mathbb{Z}^n$. The closed Lie subgroups of G are precisely the abstract subgroups of the form $\hat{K}$ where K is a subgroup of $\hat{G}$ and $\hat{K} = \{g \in G | \lambda(g) = 1 \ for \ all \ \lambda \in K\}$.

Now in the present case, Theorem 2.4 says the following: V has a decomposition into distinct (non-zero) irreducible V^G-modules of the form

$$V = \bigoplus_{\lambda \in \hat{G}} V^\lambda.$$

Then for any subgroup K of $\hat{G}$ we may define

$$V_K = \bigoplus_{\lambda \in K} V^\lambda.$$

It is clear that $V_K = V^{\hat{K}}$, and that $V_K = V_L$ if, and only if, $K = L$. So it suffices to show that each V_K is a vertex operator subalgebra of V and that every vertex operator subalgebra of V containing V^G has this form.

Now if $\lambda_1, \lambda_2 \in K < \hat{G}$ and if $u \in V^{\lambda_1}$, $v \in V^{\lambda_2}$, then certainly $Y(u, z)v$ has component vectors in $V^{\lambda_1 + \lambda_2}$. So V_K is certainly a vertex operator subalgebra of V. On the other hand if $V^G \subset W \subset V$ and W is a vertex operator subalgebra then $W = \bigoplus_{\sigma \in S} V^\sigma$ for some subset $S \subset \hat{G}$. Now as in the proof of Theorem 2.4 above (see also the proof of Theorem 2 of [DM1]) we see that S is indeed closed under multiplication and taking inverse and hence is a subgroup of $\hat{G}$. This completes the proof of Theorem 3.1.

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